Announcements

1) HW #5 due Thursday

2) Office hours 4-5 today

3) Proof that $\det(A^t) = \det(A)$ is up on yesterday's notes. Indices fixed on property 5).

Recall: $A \in M_n(\mathbb{C})$

is invertible if and only if $\det(A) \neq 0$.

Immediate Consequence

$$\boxed{\lambda \text{ is an eigenvalue for } A}$$

$$\iff A - \lambda I_n \text{ not invertible}$$

$$\iff \boxed{\det(A - \lambda I_n) = 0}$$

Definition: (characteristic polynomial)

If $A \in M_n(\mathbb{C})$, the characteristic polynomial of A is the polynomial

$$\boxed{p(\lambda) = \det(A - \lambda I_n)}$$

Note

$$\det(A-\lambda I_n) \quad (B = A-\lambda I_n)$$

$$= \det(B)$$

$$= \sum_{\sigma \in S_n} \operatorname{sign}(\sigma) \prod_{i=1}^{n} b_{i,\sigma(i)}$$

$$b_{i,\sigma(i)} = a_{i,\sigma(i)}$$

if $i \neq \sigma(i)$

If $i=\sigma(i)$,

$$b_{i,\sigma(i)} = b_{i,i}$$

$$= a_{i,i} - \lambda .$$

So the highest power of λ in the sum is n, and comes from the product

$$\prod_{i=1}^{n} (a_{ii} - \lambda) \quad (\sigma = e) .$$

This implies the characteristic polynomial of $A \in M_n(\mathbb{C})$ has degree n.

$M_n(\mathbb{C})$

By the fundamental theorem of algebra, an n^{th} degree polynomial has n roots (counted with multiplicity).

Therefore, $A \in M_n(\mathbb{C})$ has n eigenvalues (counted with multiplicity).

"counted with multiplicity"

= some values may be repeated.

i.e. If $I_n = A$, I_n has $\lambda = 1$ as the only eigenvalue, counted n times.

Example 1: $(M_2(\mathbb{R}))$

Let $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

Then $\det(A - \lambda I_2)$

$$= \det\left(\begin{bmatrix} -\lambda & 1 \\ -1 & -\lambda \end{bmatrix}\right)$$

$$= \lambda^2 + 1.$$

$\lambda^2+1=0$ if and only if

$\lambda = \pm i \notin \mathbb{R}$,

So the matrices over $\mathbb{R}$

have elements with no

eigenvalues in $\mathbb{R}$!

Theorem: (linear independence)

Let $A \in M_n(\mathbb{C})$ and let $v_1, v_2, \ldots, v_k$ $(1 \le k \le n)$ be eigenvectors corresponding to distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$, respectively.

Then $\{v_1, v_2, \ldots, v_k\}$ is linearly independent.

Proof: By contradiction:

Suppose $\exists$ scalars

$\alpha_1, \alpha_2, \dots, \alpha_k$ with

$$\sum_{i=1}^{k} \alpha_i v_i = 0.$$

This means

$$A\left(\sum_{i=1}^{k} \alpha_i v_i\right) = A(0) = 0.$$

But each v_i is an eigenvector of A with eigenvalue λ_i, hence

$$0 = A\left(\sum_{i=1}^{k} \alpha_i v_i\right)$$

$$= \sum_{i=1}^{n} \alpha_i A v_i$$

$$= \sum_{i=1}^{n} \alpha_i \lambda_i v_i$$

We then have

$$\sum_{i=1}^{k} \alpha_i \lambda_i v_i = 0$$

$$= \sum_{i=1}^{k} \alpha_i v_i .$$

This leads, since

$\lambda_i \neq \lambda_j \ \forall \ 1 \leq i, j \leq n$, to

the conclusion that

$$\alpha_1 = \alpha_2 = \cdots = \alpha_k = 0$$

The proof is by induction on k. If $k=2$, we have

$$\alpha_1 v_1 + \alpha_2 v_2 = 0$$

$$\Rightarrow \quad -\frac{\alpha_1}{\alpha_2} v_1 = v_2$$

if $\alpha_2 \neq 0$ (note $\alpha_2 = 0 \Rightarrow \alpha_1 = 0$)

Then $\lambda_2 v_2 = A v_2$

$$= A\left(-\frac{\alpha_1}{\alpha_2} v_1\right)$$

$$= \lambda_1 \left(-\frac{\alpha_1}{\alpha_2} v_1\right) = \lambda_1 v_2$$

Then $\lambda_1 = \lambda_2$, contradiction.

Now assume the result for $m = k-1$. Suppose

$$\sum_{i=1}^{k} \alpha_i v_i = 0.$$

Then as before, applying A yields

$$\sum_{i=1}^{k} \alpha_i \lambda_i v_i = 0.$$

If $\boxed{\lambda_k = 0}$, then

$\lambda_i \neq 0 \quad \forall \; 1 \leq i \leq k-1$, and

so by induction,

$$0 = \sum_{i=1}^{k} \alpha_i \lambda_i v_i = \sum_{i=1}^{k-1} \alpha_i \lambda_i v_i$$

$$\Rightarrow \alpha_i = 0 \quad \forall \; 1 \leq i \leq k-1$$

by induction.

Then

$$0 = \sum_{i=1}^{k} \alpha_i v_i = \alpha_k v_k \Rightarrow \alpha_k = 0.$$

Now assume $\boxed{\lambda_k \neq 0}$.

If $\alpha_k = 0$, then again by induction, $\alpha_1 = \alpha_2 = \cdots = \alpha_{k-1} = 0$.

So assume, in addition, that $\boxed{\alpha_k \neq 0}$.

Then we have

$$v_k = \sum_{i=1}^{k-1} -\frac{\alpha_i}{\alpha_k} v_i$$

$$= \sum_{i=1}^{k-1} -\frac{\lambda_i \alpha_i}{\lambda_k \alpha_k} v_i$$

$$\Rightarrow \sum_{i=1}^{k-1} \frac{\alpha_i}{\alpha_k} \left(1 - \frac{\lambda_i}{\lambda_k}\right) v_i = 0$$

Then by induction, for each $1 \le i \le k-1$, either

$$\frac{\alpha_i}{\alpha_k} = 0 \quad \text{or} \quad \left(1 - \frac{\lambda_i}{\lambda_k}\right) = 0.$$

But the latter implies

$\lambda_i = \lambda_k$, contradiction.

So $\dfrac{\alpha_i}{\alpha_k} = 0 \quad \forall\, 1 \le i \le k-1$

$\Rightarrow \alpha_i = 0 \quad \forall\, 1 \le i \le k-1.$

But then

$$0 = \sum_{i=1}^{k} \alpha_i v_i = \alpha_k v_k$$

$\Rightarrow \boxed{\alpha_k = 0}$, contradiction.

Then $\{v_1, v_2, \ldots, v_k\}$

must be linearly independent.

$\square$

Definition: (diagonalizability)

Let $A \in M_n(\mathbb{C})$ (or $M_n(\mathbb{R})$).

Then A is **diagonalizable**

if $\exists$ diagonal matrix

$D \in M_n(\mathbb{C})$ (or $M_n(\mathbb{R})$)

and invertible matrix

$S \in M_n(\mathbb{C})$ (or $M_n(\mathbb{R})$)

with $\boxed{A = SDS^{-1}}$

For $M_n(\mathbb{R})$, we already have seen that not every matrix is diagonalizable, since $A = SDS^{-1} \Rightarrow A$ has the same eigenvalues as D (HW #6).

Since $\exists\, A \in M_2(\mathbb{R})$ with no eigenvalues, this can't happen for all A.

Q: If $A \in M_n(\mathbb{C})$, is A diagonalizable?

A: NO!

Example 2: (non-diagonalizability)

Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \in M_2(\mathbb{C})$.

If A is diagonalizable,

$A = SDS^{-1}$, then A

has eigenvectors

$\{Se_1, Se_2\}$.

$$\det\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right)$$

$$= \det\left(\begin{bmatrix} -\lambda & 1 \\ 0 & -\lambda \end{bmatrix}\right)$$

$= \lambda^2 \Rightarrow$ the only eigenvalues of A are $\lambda = 0$ (counted twice).

So if A is similar to a diagonal matrix

$$D = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix},$$

then $\alpha = 0 = \beta$.

But then we'd have

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = S \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} S^{-1}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$